

Certain weighted sum formulas for multiple zeta values with some parameters

Shin-ya Kadota

Abstract

Multiple zeta values (MZVs) are real numbers which are defined by certain multiple series. Recently, many people have researched for relations among them and many relations are well known. In this paper, we get a new relation among them which is a generalization of a formula obtained by Eie, Liaw and Ong [1, Main Theorem], and has four parameters. Moreover, by using similar method, we also obtain a relation which has six parameters.

1 Introduction and the statement of main results

For an n -tuple of natural numbers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ($\alpha_n \geq 2$), the multiple zeta value (MZV) $\zeta(\alpha) = \zeta(\alpha_1, \alpha_2, \dots, \alpha_n)$ is defined as the following series.

$$\zeta(\alpha) = \zeta(\alpha_1, \alpha_2, \dots, \alpha_n) := \sum_{0 < \ell_1 < \ell_2 < \dots < \ell_n} \frac{1}{\ell_1^{\alpha_1} \ell_2^{\alpha_2} \dots \ell_n^{\alpha_n}}.$$

We call n the depth of $\zeta(\alpha)$, $m = \alpha_1 + \dots + \alpha_n$ the weight of $\zeta(\alpha)$. Easily one find that MZV has the following integral representation:

$$\zeta(\alpha_1, \alpha_2, \dots, \alpha_n) = \int \dots \int_{0 < t_1 < t_2 < \dots < t_m < 1} \omega_1(t_1) \omega_2(t_2) \dots \omega_m(t_m),$$

where

$$\omega_i(t) = \begin{cases} \frac{dt}{1-t} & , i \in \{1, \alpha_1 + 1, \alpha_1 + \alpha_2 + 1, \dots, \alpha_1 + \dots + \alpha_{n-1} + 1\}, \\ \frac{dt}{t} & , \text{otherwise.} \end{cases}$$

This integral representation plays an important role in this paper.

First, MZVs were studied by Euler [2] in the special case $n = 2$, and Hoffman [4] gave the above general definition. It is known that MZVs are closely related to knot theory, arithmetic geometry and mathematical physics. Zagier proposed the conjecture for the dimension of the \mathbb{Q} -vector spaces \mathcal{Z}_k spanned by MZVs with weight k by numerical calculations. According to that conjecture we can expect that there are many relations among MZVs. So in the study of MZVs, one of the main topic is to obtain various relations among them. The present paper is devoted to this topic, and the aim of the present paper is to prove the following two relations.

Let \mathfrak{S}_n be symmetric group of n -th order.

Theorem 1.1. For non-negative integers k and ℓ , and four parameters (indeterminates) μ_1, μ_2, ξ_1 and ξ_2 , we have

$$\begin{aligned}
& \sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} \mu_1^{a_1} \mu_2^{a_2} \xi_1^{b_1} \xi_2^{b_2} \zeta(a_1 + b_1 + 2) \zeta(a_2 + b_2 + 2) \\
&= \sum_{\sigma \in \mathfrak{S}_2} \left[\sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} \mu_{\sigma(1)}^{a_1} \mu_{\sigma(2)}^{a_2} \xi_{\sigma(1)}^{b_1} \xi_{\sigma(2)}^{b_2} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1, \beta_0, \beta_1, \dots, \beta_{a_2} + 1) + \right. \\
&+ \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} \mu_{\sigma(1)}^{a_1} \xi_{\sigma(1)}^{b_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\mu_{\sigma(1)}^{a_3} \xi_{\sigma(1)}^{b_3} + \mu_{\sigma(2)}^{a_3} \xi_{\sigma(2)}^{b_3}) \times \\
&\left. \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1) \right],
\end{aligned}$$

where a_i and b_i run over non-negative integers with the conditions that the sum of those are k and ℓ , and $|\alpha| := \alpha_0 + \dots + \alpha_{a_1}$, $|\beta| := \beta_0 + \dots + \beta_{a_2}$ and $|\gamma| := \gamma_0 + \dots + \gamma_{a_3}$. Therefore, for example $\sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1}}$ means that $\alpha_0, \dots, \alpha_{a_1}$ and $\beta_0, \dots, \beta_{a_2}$ run over all positive integers with $\alpha_0 + \dots + \alpha_{a_1} = a_1 + b_1 + 1$, $\beta_0 + \dots + \beta_{a_2} = a_2 + b_2 + 1$.

In **Theorem 1.1**, when k is an even number, putting $(\mu_1, \mu_2, \xi_1, \xi_2) = (1, -1, 1, 1)$ we can derive the results of Eie, Liaw and Ong [1, Main Theorem] which is a generalization of the weighted sum formula of Ohno–Zudilin [6, Theorem 3]. We will derive [1, Main Theorem] from **Theorem 1.1** in Section 3.

The next one is the main theorem in this paper.

Theorem 1.2. For non-negative integers k, ℓ and six parameters (indeterminates) $\mu_1, \mu_2, \mu_3, \xi_1, \xi_2$ and ξ_3 , we have following relation:

$$\begin{aligned}
& \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} \mu_1^{a_1} \mu_2^{a_2} \mu_3^{a_3} \xi_1^{b_1} \xi_2^{b_2} \xi_3^{b_3} \zeta(a_1 + b_1 + 2) \zeta(a_2 + b_2 + 2) \zeta(a_3 + b_3 + 2) \\
&= \sum_{\sigma \in \mathfrak{S}_3} \left[\sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} P_{1,\sigma} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1, \beta_0, \beta_1, \dots, \beta_{a_2} + 1, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1) + \right. \\
&+ \sum_{\substack{a_1+\dots+a_4=k \\ b_1+\dots+b_4=\ell}} (P_{2,\sigma} + P_{3,\sigma}) \times \\
&\times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1, \delta_0, \delta_1, \dots, \delta_{a_4} + 1) + \\
&+ \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} (P_{4,\sigma} + P_{5,\sigma} + P_{7,\sigma} + P_{12,\sigma}) \times
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1 \\ |\varepsilon|=a_5+b_5+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3}, \delta_0, \delta_1, \dots, \delta_{a_4} + \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{a_5} + 1) + \\
& + \sum_{\substack{a_1+\dots+a_4=k \\ b_1+\dots+b_4=\ell}} (P_{6,\sigma} + P_{11,\sigma}) \times \\
& \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1, \beta_0, \dots, \beta_{a_2}, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + \delta_0, \delta_1, \dots, \delta_{a_4} + 1) + \\
& + \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} (P_{8,\sigma} + P_{9,\sigma} + P_{10,\sigma} + P_{13,\sigma} + P_{14,\sigma} + P_{15,\sigma}) \times \\
& \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1 \\ |\varepsilon|=a_5+b_5+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \dots, \beta_{a_2}, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + \delta_0, \delta_1, \dots, \delta_{a_4} + \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{a_5} + 1) \Big],
\end{aligned}$$

where

$$\begin{aligned}
P_{1,\sigma} &= \mu_{\sigma(1)}^{a_1} \mu_{\sigma(2)}^{a_2} \mu_{\sigma(3)}^{a_3} \xi_{\sigma(1)}^{b_1} \xi_{\sigma(2)}^{b_2} \xi_{\sigma(3)}^{b_3}, \\
P_{2,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} \mu_{\sigma(2)}^{a_3} \mu_{\sigma(3)}^{a_4} \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} \xi_{\sigma(2)}^{b_3} \xi_{\sigma(3)}^{b_4}, \\
P_{3,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} \mu_{\sigma(3)}^{a_4} \xi_{\sigma(1)}^{b_1+b_3} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} \xi_{\sigma(3)}^{b_4}, \\
P_{4,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \xi_{\sigma(1)}^{b_1+b_3} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5}, \\
P_{5,\sigma} &= \mu_{\sigma(1)}^{a_1+a_3+a_5} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \xi_{\sigma(1)}^{b_1+b_3+b_5} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4}, \\
P_{6,\sigma} &= \mu_{\sigma(1)}^{a_1} \mu_{\sigma(2)}^{a_2} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} \mu_{\sigma(3)}^{a_4} \xi_{\sigma(1)}^{b_1} \xi_{\sigma(2)}^{b_2} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} \xi_{\sigma(3)}^{b_4}, \\
P_{7,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} \mu_{\sigma(2)}^{a_3} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} \xi_{\sigma(2)}^{b_3} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5}, \\
P_{8,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \times \\
& \times \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5}, \\
P_{9,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \times \\
& \times \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5}, \\
P_{10,\sigma} &= \mu_{\sigma(1)}^{a_1+a_5} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \times \\
& \times \xi_{\sigma(1)}^{b_1+b_5} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4}, \\
P_{11,\sigma} &= \mu_{\sigma(1)}^{a_1} \mu_{\sigma(2)}^{a_2+a_4} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} \xi_{\sigma(1)}^{b_1} \xi_{\sigma(2)}^{b_2+b_4} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3}, \\
P_{12,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} \mu_{\sigma(2)}^{a_3+a_5} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_4} \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} \xi_{\sigma(2)}^{b_3+b_5} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_4}, \\
P_{13,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} (\mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \times \\
& \times \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} (\xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(2)}^{b_5}, \\
P_{14,\sigma} &= \mu_{\sigma(1)}^{a_1} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2+a_4} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} \mu_{\sigma(2)}^{a_5} \times
\end{aligned}$$

$$\begin{aligned} & \times \xi_{\sigma(1)}^{b_1} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2+b_4} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3} \xi_{\sigma(2)}^{b_5}, \\ P_{15,\sigma} &= \mu_{\sigma(1)}^{a_1+a_5} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2+a_4} (\mu_{\sigma(1)} + \mu_{\sigma(2)} + \mu_{\sigma(3)})^{a_3} \xi_{\sigma(1)}^{b_1+b_5} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2+b_4} (\xi_{\sigma(1)} + \xi_{\sigma(2)} + \xi_{\sigma(3)})^{b_3}. \end{aligned}$$

Remark 1.3. *The formulas of Theorem 1.1 and Theorem 1.2 have several parameters, so one may get new relations by comparing the parameters of the both sides or differentiating partially with respect to parameters.*

2 Proof of Theorem 1.1 and 1.2

In this section, we prove **Theorem 1.2**. The basic structure of the proof is the same as in [1]. For $k, \ell \in \mathbb{Z}_{\geq 0}$, and 6 parameters $\mu_1, \mu_2, \mu_3, \xi_1, \xi_2$ and ξ_3 , we define $I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3)$ as the following integral:

$$\begin{aligned} & I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) \\ &:= \frac{1}{k!\ell!} \int_{\substack{0 < s_1 < s_2 < 1 \\ 0 < t_1 < t_2 < 1 \\ 0 < u_1 < u_2 < 1}} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k \times \\ & \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}. \end{aligned}$$

To obtain **Theorem 1.2**, we calculate this integral in two ways.

• The first calculations

First, we expand the factors of the integrand simply as follows:

$$\begin{aligned} & \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k \\ &= \sum_{a_1+a_2+a_3=k} \frac{k!}{a_1!a_2!a_3!} \mu_1^{a_1} \mu_2^{a_2} \mu_3^{a_3} \left(\log \frac{1-s_1}{1-s_2} \right)^{a_1} \left(\log \frac{1-t_1}{1-t_2} \right)^{a_2} \left(\log \frac{1-u_1}{1-u_2} \right)^{a_3}, \\ & \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \\ &= \sum_{b_1+b_2+b_3=\ell} \frac{\ell!}{b_1!b_2!b_3!} \xi_1^{b_1} \xi_2^{b_2} \xi_3^{b_3} \left(\log \frac{s_2}{s_1} \right)^{b_1} \left(\log \frac{t_2}{t_1} \right)^{b_2} \left(\log \frac{u_2}{u_1} \right)^{b_3}. \end{aligned}$$

Substituting these expansions, we get

$$\begin{aligned} & I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) \\ &= \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} \mu_1^{a_1} \mu_2^{a_2} \mu_3^{a_3} \xi_1^{b_1} \xi_2^{b_2} \xi_3^{b_3} \times \\ & \times \left\{ \frac{1}{a_1!b_1!} \int_{0 < s_1 < s_2 < 1} \left(\log \frac{1-s_1}{1-s_2} \right)^{a_1} \left(\log \frac{s_2}{s_1} \right)^{b_1} \frac{ds_1 ds_2}{(1-s_1)s_2} \right\} \times \end{aligned}$$

$$\begin{aligned} & \times \left\{ \frac{1}{a_2!b_2!} \int_{0 < t_1 < t_2 < 1} \left(\log \frac{1-t_1}{1-t_2} \right)^{a_2} \left(\log \frac{t_2}{t_1} \right)^{b_2} \frac{dt_1 dt_2}{(1-t_1)t_2} \right\} \times \\ & \times \left\{ \frac{1}{a_3!b_3!} \int_{0 < u_1 < u_2 < 1} \left(\log \frac{1-u_1}{1-u_2} \right)^{a_3} \left(\log \frac{u_2}{u_1} \right)^{b_3} \frac{du_1 du_2}{(1-u_1)u_2} \right\}. \end{aligned}$$

Here let us consider $\left(\log \frac{1-s_1}{1-s_2} \right)^a$. We can rewrite it as

$$\begin{aligned} \left(\log \frac{1-s_1}{1-s_2} \right)^a &= \left(\int_{s_1}^{s_2} \frac{dp}{1-p} \right)^a \\ &= \int_{s_1 < p_1 < s_2} \prod_{i=1}^a \frac{dp_i}{1-p_i} \\ &\quad \vdots \\ &\quad s_1 < p_a < s_2 \\ &= \sum_{\sigma \in \mathfrak{S}_a} \int_{s_1 < p_{\sigma(1)} < \dots < p_{\sigma(a)} < s_2} \prod_{i=1}^a \frac{dp_i}{1-p_i} \\ &= a! \int_{s_1 < p_1 < \dots < p_a < s_2} \prod_{i=1}^a \frac{dp_i}{1-p_i}. \end{aligned}$$

We can rewrite $\left(\log \frac{s_2}{s_1} \right)^b$ similarly, so we get the following lemma.

Lemma 2.1. *For $a, b \in \mathbb{Z}_{\geq 0}$, and $0 < s_1 < s_2 < 1$ we have*

$$\left(\log \frac{1-s_1}{1-s_2} \right)^a = a! \int_{s_1 < p_1 < \dots < p_a < s_2} \prod_{i=1}^a \frac{dp_i}{1-p_i}$$

and

$$\left(\log \frac{s_2}{s_1} \right)^b = b! \int_{s_1 < q_1 < \dots < q_b < s_2} \prod_{j=1}^b \frac{dq_j}{q_j},$$

where the empty product is to be understood as 1.

Using the above lemma, we get

$$\begin{aligned}
& \frac{1}{a_1!b_1!} \int_{0 < s_1 < s_2 < 1} \left(\log \frac{1-s_1}{1-s_2} \right)^{a_1} \left(\log \frac{s_2}{s_1} \right)^{b_1} \frac{ds_1 ds_2}{(1-s_1)s_2} \\
&= \int_{\substack{0 < s_1 < s_2 < 1 \\ s_1 < p_1 < \dots < p_{a_1} < s_2 \\ s_1 < q_1 < \dots < q_{b_1} < s_2}} \frac{ds_1}{1-s_1} \prod_{i=1}^{a_1} \frac{dp_i}{1-p_i} \prod_{j=1}^{b_1} \frac{dq_j}{q_j} \frac{ds_2}{s_2} \\
&= \sum_{(r_1, \dots, r_{a_1+b_1})} \int_{\substack{0 < s_1 < s_2 < 1 \\ s_1 < r_1 < \dots < r_{a_1+b_1} < s_2}} \frac{ds_1}{1-s_1} \prod_{i=1}^{a_1} \frac{dp_i}{1-p_i} \prod_{j=1}^{b_1} \frac{dq_j}{q_j} \frac{ds_2}{s_2},
\end{aligned} \tag{1}$$

where the summation runs over all tuples $(r_1, \dots, r_{a_1+b_1})$ such that

$$\{r_1, \dots, r_{a_1+b_1}\} = \{p_1, \dots, p_{a_1}\} \cup \{q_1, \dots, q_{b_1}\}$$

and

$$p_1 < \dots < p_{a_1}, q_1 < \dots < q_{b_1}.$$

Then each integral gives a multiple zeta value, which implies that the above is

$$= \sum_{|\alpha|=a_1+b_1+1} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1).$$

By using the sum formula [3, Proposition], we have

$$I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) = \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} \mu_1^{a_1} \mu_2^{a_2} \mu_3^{a_3} \xi_1^{b_1} \xi_2^{b_2} \xi_3^{b_3} \zeta(a_1+b_1+2) \zeta(a_2+b_2+2) \zeta(a_3+b_3+2).$$

This is the end of the first calculations.

• The second calculations

We divide the region $0 < s_1 < s_2 < 1, 0 < t_1 < t_2 < 1, 0 < u_1 < u_2 < 1$ to ninety regions, according to the order of magnitude of variables, and calculate the integral on each regions. But, as we will see just below, because of the symmetry of variables s, t and u , it is sufficient to calculate on the following fifteen regions:

$$\begin{aligned}
D_1 : 0 < s_1 < s_2 < t_1 < t_2 < u_1 < u_2 < 1, & D_2 : 0 < s_1 < t_1 < s_2 < t_2 < u_1 < u_2 < 1, \\
D_3 : 0 < s_1 < t_1 < t_2 < s_2 < u_1 < u_2 < 1, & D_4 : 0 < s_1 < t_1 < t_2 < u_1 < s_2 < u_2 < 1, \\
D_5 : 0 < s_1 < t_1 < t_2 < u_1 < u_2 < s_2 < 1, & \\
D_6 : 0 < s_1 < s_2 < t_1 < u_1 < t_2 < u_2 < 1, & D_7 : 0 < s_1 < t_1 < s_2 < u_1 < t_2 < u_2 < 1, \\
D_8 : 0 < s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < 1, & D_9 : 0 < s_1 < t_1 < u_1 < t_2 < s_2 < u_2 < 1, \\
D_{10} : 0 < s_1 < t_1 < u_1 < t_2 < u_2 < s_2 < 1, & \\
D_{11} : 0 < s_1 < s_2 < t_1 < u_1 < u_2 < t_2 < 1, & D_{12} : 0 < s_1 < t_1 < s_2 < u_1 < u_2 < t_2 < 1, \\
D_{13} : 0 < s_1 < t_1 < u_1 < s_2 < u_2 < t_2 < 1, & D_{14} : 0 < s_1 < t_1 < u_1 < u_2 < s_2 < t_2 < 1, \\
D_{15} : 0 < s_1 < t_1 < u_1 < u_2 < t_2 < s_2 < 1. &
\end{aligned}$$

We define $I_{k,\ell,m}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3)$ as the integral on the region D_m ($m = 1, \dots, 15$):

$$\begin{aligned} & I_{k,\ell,m}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) \\ &:= \frac{1}{k!\ell!} \int_{D_m} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k \times \\ & \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2}. \end{aligned}$$

How to treat the remaining seventy five regions? For example, we can see that the integration on the region $0 < u_1 < u_2 < s_1 < s_2 < t_1 < t_2 < 1$ which is one of the remaining seventy five regions is written by using the integration on D_1 as follows:

$$\begin{aligned} & \frac{1}{k!\ell!} \int_{0 < u_1 < u_2 < s_1 < s_2 < t_1 < t_2 < 1} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} + \mu_3 \log \frac{1-u_1}{1-u_2} \right)^k \times \\ & \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} + \xi_3 \log \frac{u_2}{u_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2 du_1 du_2}{(1-s_1)s_2(1-t_1)t_2(1-u_1)u_2} \\ & = I_{k,\ell,1}(\mu_3, \mu_1, \mu_2, \xi_3, \xi_1, \xi_2). \end{aligned}$$

Therefore the integrals on the remaining seventy five regions are obtained by changing parameters in the integrals on D_1, D_2, \dots, D_{14} or D_{15} . After all, we obtain

$$I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) = \sum_{\sigma \in \mathfrak{S}_3} \sum_{m=1}^{15} I_{k,\ell,m}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}).$$

To save pages, we present only the calculations on the region $D_4 : 0 < s_1 < t_1 < t_2 < u_1 < s_2 < u_2 < 1$ in this paper. We want to write the integral as an explicit sum of MZVs, hence we need to modify some terms in the integrand before we expand the integrand. Those modifications are the most important point in this proof. We expand the integrand as follows:

$$\begin{aligned} & \left(\mu_{\sigma(1)} \log \frac{1-s_1}{1-s_2} + \mu_{\sigma(2)} \log \frac{1-t_1}{1-t_2} + \mu_{\sigma(3)} \log \frac{1-u_1}{1-u_2} \right)^k \\ &= \left\{ \mu_{\sigma(1)} \log \frac{1-s_1}{1-t_1} + (\mu_{\sigma(1)} + \mu_{\sigma(2)}) \log \frac{1-t_1}{1-t_2} + \mu_{\sigma(1)} \log \frac{1-t_2}{1-u_1} + \right. \\ & \quad \left. + (\mu_{\sigma(1)} + \mu_{\sigma(3)}) \log \frac{1-u_1}{1-s_2} + \mu_{\sigma(3)} \log \frac{1-s_2}{1-u_2} \right\}^k \tag{2} \\ &= \sum_{a_1+a_2+a_3+a_4+a_5=k} \frac{k!}{a_1!a_2!a_3!a_4!a_5!} \mu_{\sigma(1)}^{a_1+a_3} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \times \\ & \quad \times \left(\log \frac{1-s_1}{1-t_1} \right)^{a_1} \left(\log \frac{1-t_1}{1-t_2} \right)^{a_2} \left(\log \frac{1-t_2}{1-u_1} \right)^{a_3} \left(\log \frac{1-u_1}{1-s_2} \right)^{a_4} \left(\log \frac{1-s_2}{1-u_2} \right)^{a_5}, \end{aligned}$$

and

$$\begin{aligned}
& \left(\xi_{\sigma(1)} \log \frac{s_2}{s_1} + \xi_{\sigma(2)} \log \frac{t_2}{t_1} + \xi_{\sigma(3)} \log \frac{u_2}{u_1} \right)^\ell \\
&= \left\{ \xi_{\sigma(1)} \log \frac{t_1}{s_1} + (\xi_{\sigma(1)} + \xi_{\sigma(2)}) \log \frac{t_2}{t_1} + \xi_{\sigma(1)} \log \frac{u_1}{t_2} + \right. \\
&\quad \left. + (\xi_{\sigma(1)} + \xi_{\sigma(3)}) \log \frac{s_2}{u_1} + \xi_{\sigma(3)} \log \frac{u_2}{s_2} \right\}^\ell \tag{3} \\
&= \sum_{b_1+b_2+b_3+b_4+b_5=\ell} \frac{\ell!}{b_1!b_2!b_3!b_4!b_5!} \xi_{\sigma(1)}^{b_1+b_3} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5} \times \\
&\quad \times \left(\log \frac{t_1}{s_1} \right)^{b_1} \left(\log \frac{t_2}{t_1} \right)^{b_2} \left(\log \frac{u_1}{t_2} \right)^{b_3} \left(\log \frac{s_2}{u_1} \right)^{b_4} \left(\log \frac{u_2}{s_2} \right)^{b_5}.
\end{aligned}$$

The above modifications are based on the following observation. First consider $\log \frac{1-s_1}{1-s_2}$. In this case, there are t_1, t_2 and u_1 between s_1 and s_2 , so we modify $\log \frac{1-s_1}{1-s_2}$ to the following form.

$$\begin{aligned}
\log \frac{1-s_1}{1-s_2} &= \int_{s_1}^{s_2} \frac{dp}{1-p} \\
&= \int_{s_1}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{u_1} + \int_{u_1}^{s_2} \frac{dp}{1-p} \\
&= \log \frac{1-s_1}{1-t_1} + \log \frac{1-t_1}{1-t_2} + \log \frac{1-t_2}{1-u_1} + \log \frac{1-u_1}{1-s_2}.
\end{aligned}$$

There is s_2 between u_1 and u_2 , so we modify $\log \frac{1-u_1}{1-u_2}$ similarly:

$$\begin{aligned}
\log \frac{1-u_1}{1-u_2} &= \int_{u_1}^{u_2} \frac{dp}{1-p} \\
&= \int_{u_1}^{s_2} + \int_{s_2}^{u_2} \frac{dp}{1-p} \\
&= \log \frac{1-u_1}{1-s_2} + \log \frac{1-s_2}{1-u_2}.
\end{aligned}$$

These modifications give (2), and similarly we can show (3). Substituting the above modified expansions and using **Lemma 2.1** and arguing in the same way as (1), we get

$$\begin{aligned}
& I_{k,\ell,4}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}) \\
&= \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} P_{4,\sigma} \int_{D'_4} \frac{ds_1}{1-s_1} \prod_{i=1}^{a_1} \frac{dp_i}{1-p_i} \prod_{j=1}^{b_1} \frac{dq_j}{q_j} \frac{dt_1}{1-t_1} \prod_{i=a_1+1}^{a_1+a_2} \frac{dp_i}{1-p_i} \prod_{j=b_1+1}^{b_1+b_2} \frac{dq_j}{q_j} \frac{dt_2}{t_2} \times \\
&\quad \times \prod_{i=a_1+a_2+1}^{a_1+a_2+a_3} \frac{dp_i}{1-p_i} \prod_{j=b_1+b_2+1}^{b_1+b_2+b_3} \frac{dq_j}{q_j} \frac{du_1}{1-u_1} \prod_{i=a_1+a_2+a_3+1}^{a_1+\dots+a_4} \frac{dp_i}{1-p_i} \prod_{j=b_1+b_2+b_3+1}^{b_1+\dots+b_4} \frac{dq_j}{q_j} \frac{ds_2}{s_2} \times \\
&\quad \times \prod_{i=a_1+\dots+a_4+1}^{a_1+\dots+a_5} \frac{dp_i}{1-p_i} \prod_{j=b_1+\dots+b_4+1}^{b_1+\dots+b_5} \frac{dq_j}{q_j} \frac{du_2}{u_2}
\end{aligned}$$

$$= \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} P_{4,\sigma} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1 \\ |\varepsilon|=a_5+b_5+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3},$$

$$\delta_0, \delta_1, \dots, \delta_{a_4} + \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{a_5} + 1),$$

where

$$D'_4 = \left\{ (s_1, s_2, t_1, t_2, u_1, u_2, p_1, \dots, p_k, q_1, \dots, q_\ell) \in (0, 1)^{k+\ell+4} \mid \begin{array}{l} 0 < s_1 < t_1 < t_2 < u_1 < s_2 < u_2 < 1 \\ s_1 < p_1 < \dots < p_{a_1} < t_1 \\ s_1 < q_1 < \dots < q_{b_1} < t_1 \\ t_1 < p_{a_1+1} < \dots < p_{a_1+a_2} < t_2 \\ t_1 < q_{b_1+1} < \dots < q_{b_1+b_2} < t_2 \\ t_2 < p_{a_1+a_2+1} < \dots < p_{a_1+a_2+a_3} < u_1 \\ t_2 < q_{b_1+b_2+1} < \dots < q_{b_1+b_2+b_3} < u_1 \\ u_1 < p_{a_1+a_2+a_3+1} < \dots < p_{a_1+\dots+a_4} < s_2 \\ u_1 < q_{b_1+b_2+b_3+1} < \dots < q_{b_1+\dots+b_4} < s_2 \\ s_2 < p_{a_1+\dots+a_4+1} < \dots < p_{a_1+\dots+a_5} < u_2 \\ s_2 < q_{b_1+\dots+b_4+1} < \dots < q_{b_1+\dots+b_5} < u_2 \end{array} \right\}$$

and

$$P_{4,\sigma} = \mu_{\sigma(1)}^{a_1+a_3} (\mu_{\sigma(1)} + \mu_{\sigma(2)})^{a_2} (\mu_{\sigma(1)} + \mu_{\sigma(3)})^{a_4} \mu_{\sigma(3)}^{a_5} \xi_{\sigma(1)}^{b_1+b_3} (\xi_{\sigma(1)} + \xi_{\sigma(2)})^{b_2} (\xi_{\sigma(1)} + \xi_{\sigma(3)})^{b_4} \xi_{\sigma(3)}^{b_5}.$$

This is the end of the calculations for $I_{k,\ell,4}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)})$. When one calculate the other $I_{k,\ell,m}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)})$ by the same way of modifications, then the $P_{m,\sigma}$ which is in **Theorem 1.2** appears as the coefficient of modified integrand for the integral of D_m . Namely, one obtain

$$I_{k,\ell,m}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \mu_{\sigma(3)}, \xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}) = \sum_{a,b} P_{m,\sigma} \times (\text{the sum of MZVs})$$

as above. Moreover, several types of the sum of MZVs will appear in the calculations, but one can notice that these types of the sum of MZVs depend only on the subscript of divided regions. Namely, the sum of MZVs which appears after the calculations for D_2 and D_3 are the same, for D_4, D_5, D_7 and D_{12} are the same, for D_6 and D_{11} are the same, and for $D_8, D_9, D_{10}, D_{13}, D_{14}$ and D_{15} are the same. Noting this point, we obtain

$$\begin{aligned} & I_{k,\ell}(\mu_1, \mu_2, \mu_3, \xi_1, \xi_2, \xi_3) \\ &= \sum_{\sigma \in \mathfrak{S}_3} \left[\sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} P_{1,\sigma} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1, \beta_0, \beta_1, \dots, \beta_{a_2} + 1, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1) + \right. \\ &+ \sum_{\substack{a_1+\dots+a_4=k \\ b_1+\dots+b_4=\ell}} (P_{2,\sigma} + P_{3,\sigma}) \times \\ &\times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1, \delta_0, \delta_1, \dots, \delta_{a_4} + 1) + \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} (P_{4,\sigma} + P_{5,\sigma} + P_{7,\sigma} + P_{12,\sigma}) \times \\
& \quad \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1 \\ |\varepsilon|=a_5+b_5+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3}, \delta_0, \delta_1, \dots, \delta_{a_4} + \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{a_5} + 1) + \\
& + \sum_{\substack{a_1+\dots+a_4=k \\ b_1+\dots+b_4=\ell}} (P_{6,\sigma} + P_{11,\sigma}) \times \\
& \quad \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1} + 1, \beta_0, \dots, \beta_{a_2}, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + \delta_0, \delta_1, \dots, \delta_{a_4} + 1) + \\
& + \sum_{\substack{a_1+\dots+a_5=k \\ b_1+\dots+b_5=\ell}} (P_{8,\sigma} + P_{9,\sigma} + P_{10,\sigma} + P_{13,\sigma} + P_{14,\sigma} + P_{15,\sigma}) \times \\
& \quad \times \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\gamma|=a_3+b_3+1 \\ |\delta|=a_4+b_4+1 \\ |\varepsilon|=a_5+b_5+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \dots, \beta_{a_2}, \gamma_0, \gamma_1, \dots, \gamma_{a_3} + \delta_0, \delta_1, \dots, \delta_{a_4} + \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{a_5} + 1) \Big].
\end{aligned}$$

Combining the conclusions of the first calculations and the second calculations, we obtain the asserted relations of **Theorem 1.2**.

To prove **Theorem 1.1**, it is sufficient to calculate similarly the following integrals:

$$\begin{aligned}
& I_{k,\ell}(\mu_1, \mu_2, \xi_1, \xi_2) \\
& := \frac{1}{k!\ell!} \int_{\substack{0 < s_1 < s_2 < 1 \\ 0 < t_1 < t_2 < 1}} \left(\mu_1 \log \frac{1-s_1}{1-s_2} + \mu_2 \log \frac{1-t_1}{1-t_2} \right)^k \times \\
& \quad \times \left(\xi_1 \log \frac{s_2}{s_1} + \xi_2 \log \frac{t_2}{t_1} \right)^\ell \frac{ds_1 ds_2 dt_1 dt_2}{(1-s_1)s_2(1-t_1)t_2}.
\end{aligned}$$

We omit the details of the proof.

3 Deduction of the result of Eie, Liaw and Ong, and some more generalizations

Let us consider a special case of **Theorem 1.1**. Putting $\xi_1 = \xi_2 = \xi \neq 0$ in the formula of **Theorem 1.1**, and using the harmonic product $\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a+b)$ for the left

hand side, we find that ξ^ℓ parts of the both sides are cancelled with each other, and

$$\begin{aligned}
& (\ell+1)\zeta(k+\ell+4) \sum_{a_1+a_2=k} \mu_1^{a_1} \mu_2^{a_2} + \sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} (\mu_1^{a_1} \mu_2^{a_2} + \mu_2^{a_1} \mu_1^{a_2}) \zeta(a_2+b_2+2, a_1+b_1+2) \\
&= \sum_{\substack{a_1+a_2=k \\ b_1+b_2=\ell}} (\mu_1^{a_1} \mu_2^{a_2} + \mu_2^{a_1} \mu_1^{a_2}) \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1}+1, \beta_0, \beta_1, \dots, \beta_{a_2}+1) + \\
&+ \sum_{\substack{a_1+a_2+a_3=k \\ b_1+b_2+b_3=\ell}} (\mu_1^{a_1} + \mu_2^{a_1})(\mu_1 + \mu_2)^{a_2} 2^{b_2} (\mu_1^{a_3} + \mu_2^{a_3}) \\
&\sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\delta|=a_3+b_3+1}} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1).
\end{aligned} \tag{4}$$

Applying Ohno's relation ([5]) for the index $(\underbrace{1, \dots, 1}_{a_1}, 2, \underbrace{1, \dots, 1}_{a_2}, 2)$, we can see that

$$\begin{aligned}
& \sum_{b_1+b_2=\ell} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1}} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{a_1}+1, \beta_0, \beta_1, \dots, \beta_{a_2}+1) \\
&= \sum_{b_1+b_2=\ell} \zeta(a_2+b_2+2, a_1+b_1+2).
\end{aligned}$$

Then the first term on the right hand side of (4) is same as the the second term on the left hand side of (4). The second term on the right hand side of (4) can be written as follows.

$$\begin{aligned}
& \sum_{b_1+b_2+b_3=\ell} \sum_{\substack{|\alpha|=a_1+b_1+1 \\ |\beta|=a_2+b_2+1 \\ |\delta|=a_3+b_3+1}} 2^{b_2} \zeta(\alpha_0, \dots, \alpha_{a_1}, \beta_0, \beta_1, \dots, \beta_{a_2} + \gamma_0, \gamma_1, \dots, \gamma_{a_3} + 1) \\
&= \sum_{|\alpha|=k+\ell+3} 2^{\alpha_{a_1}+1+\dots+\alpha_{a_1+a_2+1}-1-a_2} (1 - 2^{1-\alpha_{a_1+a_2+1}}) \zeta(\alpha_0, \alpha_1, \dots, \alpha_{k+1} + 1).
\end{aligned}$$

Then we have

$$\begin{aligned}
& \sum_{|\alpha|=k+\ell+3} \sum_{a_1+a_2+a_3=k} (\mu_1^{a_1} + \mu_2^{a_1})(\mu_1 + \mu_2)^{a_2} (\mu_1^{a_3} + \mu_2^{a_3}) \times \\
& \times 2^{\alpha_{a_1}+1+\dots+\alpha_{a_1+a_2+1}-1-a_2} (1 - 2^{1-\alpha_{a_1+a_2+1}}) \zeta(\alpha_1, \alpha_1, \dots, \alpha_{k+1} + 1) \\
&= (\ell+1)\zeta(k+\ell+4) \sum_{a_1+a_2=k} \mu_1^{a_1} \mu_2^{a_2}.
\end{aligned}$$

In particular, when $\mu_1 = -\mu_2 = \mu \neq 0$ and k is even, the right hand side is

$$\mu^k (\ell+1) \zeta(k+\ell+4),$$

and the left hand side is

$$\mu^k \left\{ \sum_{|\alpha|=k+\ell+3} \sum_{j=0}^{\frac{k}{2}} 2^{\alpha_{2j+1}+1} \zeta(\alpha_0, \alpha_1, \dots, \alpha_{k+1}+1) - 2(k+2)\zeta(k+\ell+4) \right\}.$$

Therefore we have

$$\sum_{|\alpha|=k+\ell+3} \sum_{j=1}^{\frac{k}{2}+1} 2^{\alpha_{2j+1}+1} \zeta(\alpha_1, \alpha_2, \dots, \alpha_{k+2}+1) = (2k+\ell+5)\zeta(k+\ell+4).$$

For positive integers m, n with $m > 2n$, setting $k = 2n - 2$, $\ell = m - k - 3$, we get [1, Main Theorem]. When $m = 2n$, [1, Main Theorem] is derived from the duality theorem. Therefore we now conclude that [1, Main Theorem] can be deduced from our **Theorem 1.1**.

Remark 3.1. *More generally, by calculating integrals*

$$\begin{aligned} & I_{k,\ell}(\mu_1, \dots, \mu_n, \xi_1, \dots, \xi_n) \\ &= \frac{1}{k!\ell!} \int_{\substack{0 < x_1 < y_1 < 1 \\ \vdots \\ 0 < x_n < y_n < 1}} \left(\sum_{i=1}^n \mu_i \log \frac{1-x_i}{1-y_i} \right)^k \left(\sum_{j=1}^n \xi_j \log \frac{y_j}{x_j} \right)^\ell \prod_{h=1}^n \frac{dx_h dy_h}{(1-x_h)y_h}, \end{aligned}$$

we could get the same type of relations, but it seems difficult to write down the general form explicitly.

References

- [1] M. Eie, W. -C. Liaw, Y. L. Ong, *On generalizations of weighted sum formulas of multiple zeta values*, Int. J. Number Theory **9** (2013), no. 5, 1185–1198.
- [2] L. Euler, *Meditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol **20** (1776), 140–186; reprinted in Opera Omnia, Series I, Vol. 15, B. G. Teubner, Berlin (1927), 217–267.
- [3] A. Granville, *A decomposition of Riemann’s zeta-function*, in “Analytic number theory” London Math. Soc. Lecture Note Ser., **247**, Cambridge, 1997, pp.95–101.
- [4] M. E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), no. 2, 275–290.
- [5] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory **74** (1999), no. 1, 39–43.
- [6] Y. Ohno, W. Zudilin, *Zeta stars*, Commun. Number Theory Phys. **2** (2008), no. 2, 325–347.